Anomalous height fluctuation width in crossover from random to coherent surface growths

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We study the anomalous behavior of the height fluctuation width in the crossover from random to coherent growth of a surface for a stochastic model. In the model, random numbers are assigned on perimeter sites of a surface, representing pinning strengths of disordered media. At each time step the surface is advanced at the site having a minimum pinning strength in a random subset of the system rather than at a global minimum. The subset is composed of a randomly selected site and its $(l-1)$ neighbors. The height fluctuation width $W^2(L;l)$ exhibits nonmonotonic behavior with l and has a minimum at l^*. It is found numerically that l^* scales as $l^* \sim L^{0.59}$ and the height fluctuation width at that minimum, $W^2(L;l^*)$, scales as $\sim L^{0.85}$ in 1+1 dimensions. It is also found that the subset size $l^*(L)$ is the characteristic size of the crossover from the random surface growth in the Kardar-Parisi-Zhang universality to the coherent surface growth in the directed percolation universality. [S1063-651X(97)01903-X]

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The field of nonequilibrium surface growth has been of interest for the theoretical classification of universality and also for applications to various physical phenomena such as crystal growth, molecular-beam epitaxy, vapor deposition, biological evolution [1]. An interesting feature of nonequilibrium surface growth is the nontrivial scaling behavior of the height fluctuation width [2], i.e.,

$$W^2(L,t)=\left\langle \frac{1}{L^{d'}}\sum_i (h_i-\overline{h})^2 \right\rangle \sim L^{2\alpha} f(t/L^z), \quad (1)$$

where h_i is the height of site i on a substrate. Here $\overline{h}$, L, and d' denote the mean height, system size, and substrate dimension, respectively. The angular brackets stand for the statistical average. The scaling function behaves as $f(x)\to$const for $x \gg 1$, and $f(x)\sim x^{2\beta}$ for $x \ll 1$ with $\beta=\alpha/z$. The exponents α, β, and z are called the roughness exponent, the growth exponent, and the dynamic exponent, respectively.

The Kardar-Parisi-Zhang (KPZ) equation [3] was introduced to account for the effect of sideways growth, which is written as

$$\frac{\partial h(x,t)}{\partial t}=\nu \nabla^2 h(x,t)+\frac{\lambda}{2}[\nabla h(x,t)]^2+\eta(x,t), \quad (2)$$

where $\eta(x,t)$, the thermal noise, is assumed to be white noise, $\langle \eta(x,t)\rangle=0$; and $\langle \eta(x,t)\eta(x',t')\rangle=2D\delta^{d'}(x-x')\delta(t-t')$ with noise strength D. Many stochastic models in the KPZ universality class have been introduced [1]. Among them is the restricted solid-on-solid (RSOS) model, which was introduced by Kim and Kosterlitz [4], satisfying the scaling relation $\alpha_t+z_t=2$. The subscript means that the exponents are for *thermal* noise. In the RSOS model, a particle is deposited at a randomly selected site as long as the height difference Δh between nearest-neighbor columns remains as $\Delta h \leq 1$ even after deposition; otherwise, the particle is excluded from deposition. On the other hand, one may modify the dynamic rule of the RSOS model by replacing the exclusion rule with an avalanche rule: At each time step, a particle is deposited at a randomly selected site and an avalanche may occur successively at nearest-neighbor sites as long as the height difference between nearest neighbors $\Delta h>1$. Then it was argued that this model also belongs to the KPZ universality class [5], but the model requires a relatively large system size to see its asymptotic behavior. Let us call the former model the RSOS *A* model and the latter model the RSOS *B* model. The surface growth in the KPZ universality class is called the random surface growth.

Physical properties of the growing surface in disordered media are different from those of the thermal KPZ equation, Eq. (2). In order to account for the effect of disorder in porous media, quenched noise [6], which depends on position x and height h, replaces thermal noise in Eq. (2). Then the quenched KPZ (QKPZ) equation is written as

$$\frac{\partial h(x,t)}{\partial t}=\nu \nabla^2 h(x,t)+\frac{\lambda}{2}[\nabla h(x,t)]^2+\eta(x,h), \quad (3)$$

where the noise satisfies $\langle \eta(x,h)\rangle=0$ and $\langle \eta(x,h)\eta(x',h')\rangle=2D\delta^{d'}(x-x')\delta(h-h')$. Stochastic models associated with the QKPZ equation have been introduced [7,8]. The models show that the surface of the QKPZ equation in 1+1 dimensions belongs to the directed percolation (DP) universality. The roughness exponent α_q in the QKPZ equation is given by the ratio of the correlation length exponents in the perpendicular and parallel directions $\nu_\perp$ and $\nu_\parallel$ of directed percolating clusters, which is $\alpha_q=\nu_\perp/\nu_\parallel \approx 0.63$. On the other hand, recently Sneppen introduced a stochastic model in which the surface grows coherently [5]. In that model, random numbers, representing the disorder of porous media, are assigned to each perimeter site of the surface. The surface is advanced at the site having a global minimum among the random numbers. The avalanche rule is then applied successively to nearest-neighbor sites as long as $\Delta h>1$. Random numbers at the columns with increased heights are updated by new ones. The

Sneppen model also belongs to the DP universality class and the roughness exponent is $\alpha_s \approx 0.63$ in 1+1 dimensions [9]. The surface growth of the Sneppen model is called coherent surface growth.

The coherent surface-growth model is closely related to the self-organized evolution model for biological systems [10]. In the evolution model, one considers random numbers assigned in a one-dimensional array, the numbers representing the fitness of each species. The mutation of species is described in the model by updating random numbers. The updating occurs at the site having a global minimum random number and its two nearest neighbors, and the random numbers at those sites are replaced by new ones at each time. Then as times go on, relatively small random numbers disappear by the updating process and the distribution of random numbers exhibits a self-organized critical behavior. On the other hand, one may think of the situation where biological evolution is not motivated in a globally optimized manner, but it may be driven by optimization within the *finite* region out of the entire system. This is analogous to the case that the spin-glass system is in a metastable state within a finite relaxation time rather than in a globally stable state. Motivated by this idea, in this paper we introduce a surface growth model in disordered media, where surface growth occurs at the site having the minimum random number in a *subset* of the entire system rather than having the global minimum random number. We think that this model might be relevant to the case where the relaxation of surface growth in disordered media is not fast enough to spread into the whole system, so that the surface growth is driven not in a globally optimized manner, but in a locally optimized manner.

To be specific, the model we consider in this paper is defined as follows. First, we consider a one-dimensional flat substrate with system size L. Random numbers are assigned on each site, the numbers representing energy barriers (fitness) in the evolution model or pinning forces in the surface growth model by Sneppen. Second, we select a site randomly and consider the subset composed of $l \equiv 2r+1$ elements, the randomly selected site, and its $2r$ neighbor sites within distance r. It is worthwhile to note that the subset is regarded as a random sample because the site in the middle of the subset was selected at random. The subset is formed instantaneously and its territory might overlap with a subsequent one as shown in Fig. 1. Next, the surface is advanced at the site having a minimum random number among the l elements in the subset. The avalanche process is followed successively at its neighboring sites to keep the RSOS condition $\Delta h \leqslant 1$, and it may spread out over the boundary of the subset. Finally, the random numbers at the sites with increased height are updated with new ones. The dynamic rule of the model is depicted in Fig. 1. When $l=1$, this model corresponds to the RSOS B model in the KPZ universality class, whereas when $l=L$, it corresponds to the Sneppen B model in the DP universality class. Accordingly, one may see the crossover behavior from the KPZ limit to the DP limit with increasing l.

Since the roughness exponent $\alpha_t = 1/2$ in the KPZ limit is smaller than the one $\alpha_q \approx 0.63$ in the DP limit, one may expect at a glance that the height fluctuation width $W^2(L;l)$ in a steady state increases monotonically with increasing l. However, we found numerically the following anomalous behavior $W^2(L;l)$ decreases with increasing l for small l and increases for large l as shown in Fig. 2. The minimum of $W^2(L;l)$ becomes steeper and its location l^*/L, which was rescaled by system size L, approaches zero as L increases. It is found that the location of the minimum scales as $l^* \sim L^{0.59}$ and the height fluctuation width at this minimum scales as $W^2(L;l^*) \sim L^{0.85}$, which are shown in Figs. 3 and 4. The estimated values of l^* and $W^2(L;l^*)$ for different system sizes are tabulated in Table I.

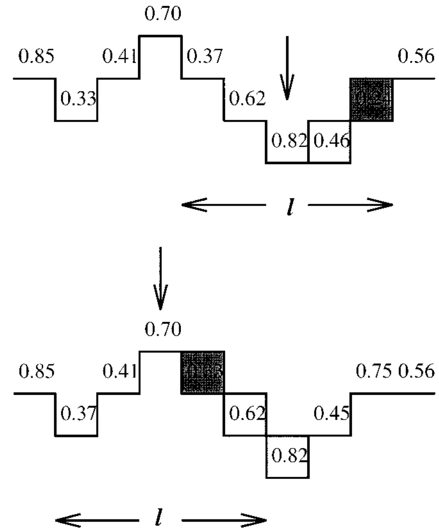

FIG. 1. Schematic representation of the stochastic rule. The arrowed sites are randomly selected sites. The dark squares denote the sites of minimum random number within the subset of size $l=5$. The white squares denote the sites updated by avalanches. The subset could overlap with the subsequent (lower) one.

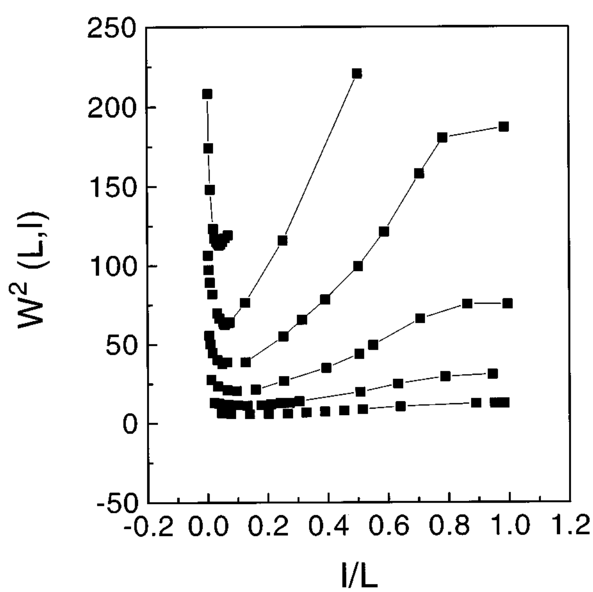

FIG. 2. Plot of the surface height fluctuation widths $W^2(L;l)$ versus subset size l/L, rescaled by system size L in a steady state. The numerical data are for system sizes L=64, 128, 256, 512, 1024, and 2048, from bottom to top.

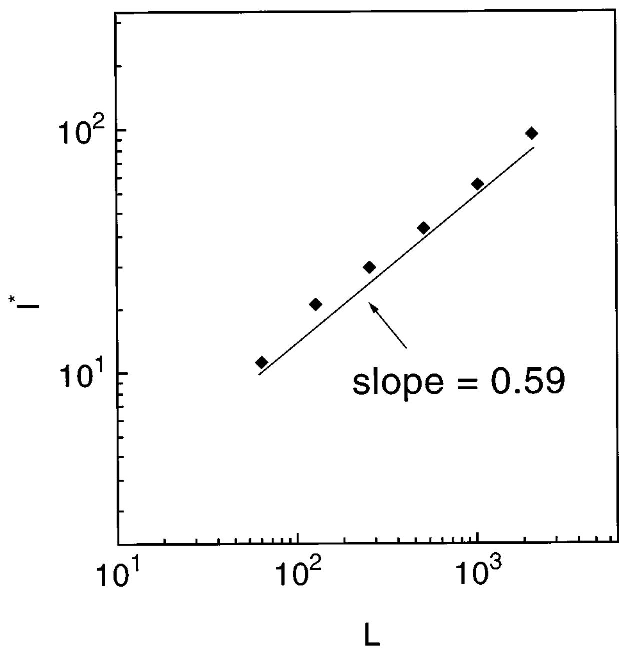

FIG. 3. Double-logarithmic plot of the estimated location l^* of the minimum versus system size L. The data are for $L=64$, 128, 256, 512, 1024, and 2048.

ôThe anomalous behavior may be attributed to two effects: the random effect for small l and the coherent effect for large l. For small l, the surface grows by random deposition with avalanches and belongs to the KPZ universality. Thus the height fluctuation width depends on the system size as $W^2(L;l) \sim L^{2\alpha_t}$, with $2\alpha_t=1$, however, it would also depend on the subset size l. In order to find the l-dependent behavior of $W^2(L;l)$ on a phenomenological level, we plot $W^2(L;l)$ versus l in double-logarithmic scales for several

TABLE I. Numerical values of the location l^* and the height fluctuation width $W^2(L;l^*)$ at the minimum for different system sizes.

L	l^*	$W^2(L;l^*)$
64	11	5.9
128	19	11.2
256	27	20.3
512	39	35.8
1024	59	62.5
2048	95	110.0

values of L as shown in Fig. 5. The slopes are measured to be ≈ -0.2 for large system sizes $L=1024$ and 2048. Based on this measurement, $W^2(L;l)$ is written as $W^2(L;l) \sim l^{-0.2}L$ for small l. This result is contrary to the one based on the coarse-graining scaling argument $W^2(L;l) \sim (L/l)^{2\alpha_t} l^{2\alpha_q}$, which exhibits an increasing behavior of W^2 with increasing l. On the other hand, when l is large enough, surface updating is initiated mainly at the site having the global minimum random number of the entire system. When the site of the global minimum is selected, which occurs with probability l/L, the surface becomes correlated by the Sneppen dynamics; however, when the site of the global minimum is not selected, the correlation formed by Sneppen's dynamics is relaxed. Thus, when l is large enough so that the contribution by the Sneppen dynamics is sufficiently dominant, one may write the dominant term of the *averaged* height fluctuation width as $W^2(L;l) \sim (l/L) L^{2\alpha_q}$. That is because the statistical average was taken over the quantity of the square of mean-height deviation in Eq. (1). Combining the two asymptotic behaviors obtained on a phenomenological level, the height fluctuation width is written as

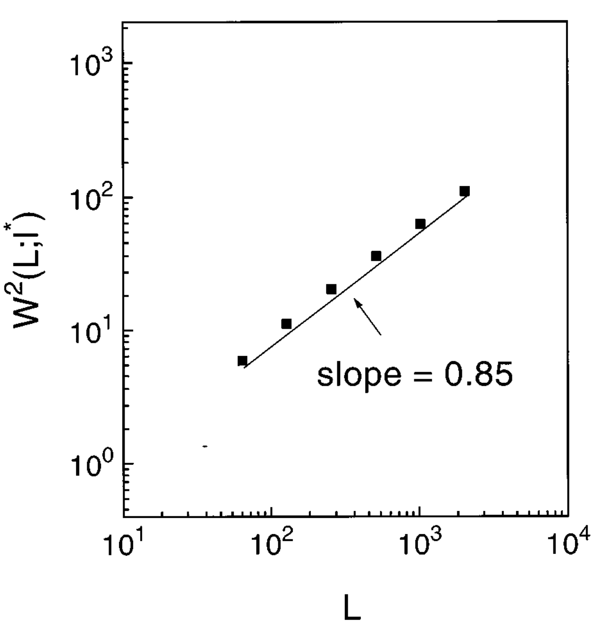

FIG. 4. Double-logarithmic plot of the surface height fluctuation width at the minimum position $W^2(L;l^*)$ versus system size L. The data are for $L=64$, 128, 256, 512, 1024, and 2048.

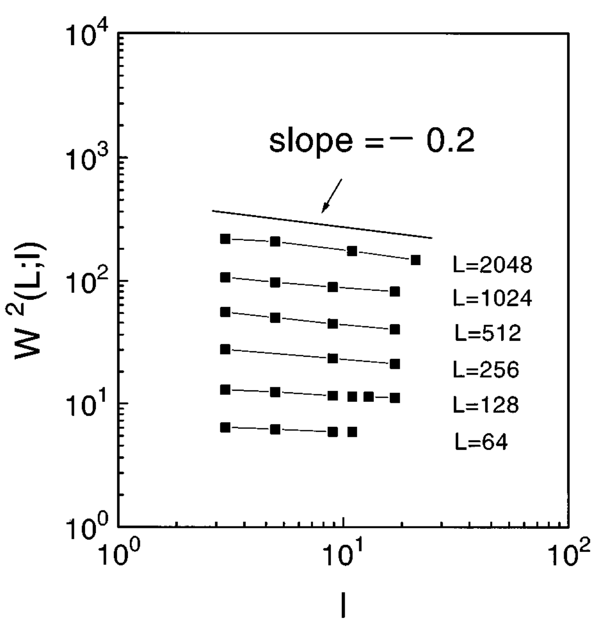

FIG. 5. Double-logarithmic plot of $W^2(L;l)$ versus l for various system sizes. The data seem to be on straight lines with slope -0.2.

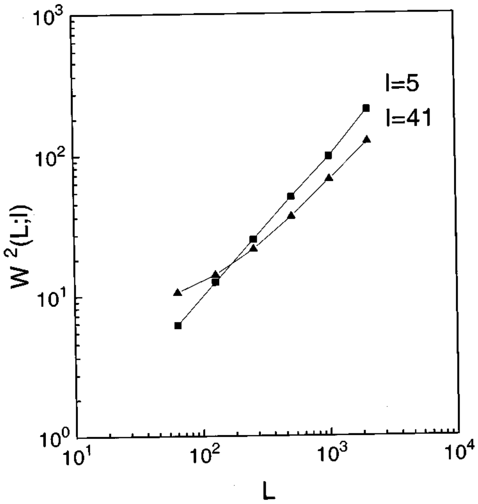

FIG. 6. Double-logarithmic plot of $W^2(L;l)$ versus L for typical subset sizes $l=5$ and 41.

$$W^2(L;l) \sim l^{-0.2}L + \left(\frac{l}{L}\right) L^{1.26}. \tag{4}$$

The two terms exhibit competing behavior with respect to l, and this yields the anomalous behavior. Taking the derivative with respect to l, the location l^* of the minimum is obtained as $l^* \sim L^{0.62}$, which is close to the numerical measurement $l^* \sim L^{0.59}$. We also examine the size-dependent behavior of the height fluctuation width at the minimum by plugging l^* into Eq. (4) and obtain that $W^2(L;l^*) \sim L^{0.88}$. This result is also close to the numerical measurement $W^2(L;l^*) \sim L^{0.85}$. The numerical estimations for l^* and $W^2(L;l^*)$ are better explained by minimizing the formula

$$W^2(L;l) \sim \left(\frac{l^{2\alpha_t}}{l^{2\alpha_q}}\right) L^{2\alpha_t} + \left(\frac{l}{L}\right) L^{2\alpha_q}; \tag{5}$$

however, the derivation of this formula is not clear.

In order to understand the physical meaning of the characteristic subset size l^*, we plot $W^2(L;l)$ versus L up to $L=2048$ in double-logarithmic scales for typical subset sizes $l=5$ and 41 in Fig. 6. The size $l=5$ corresponds to the case where it is smaller than $l^*(L)$ for all system sizes L used in Fig. 6. However, the size of $l=41$ corresponds to the case where it is smaller than $l^*(L)$ in part for $L=1024$ and 2048, close to $l^*(L)$ for $L=512$, but larger than $l^*(L)$ in part for $L=64$ and 128. For the case $l=5$, all data are on a straight line, whereas for the case $l=41$, forming a straight line breaks down for smaller system sizes $L=64$ and 128. Figure 6 suggests that $l^*(L)$ be the characteristic subset size such that when $l<l^*(L)$, the roughness of the overall surface is determined by a random effect, whereas when $l>l^*(L)$, it is done by a coherent effect. Accordingly, the subset size l^* has the meanings of not only the location of the minimum of the anomalous height fluctuation width, but also the critical size at which the crossover from random to coherent surface growths occurs. Also note that even for the case $l>l^*(L)$, the surface height fluctuation width does not behave as $W^2(L;l) \sim L^{2\alpha_q}$. The roughness exponent has a

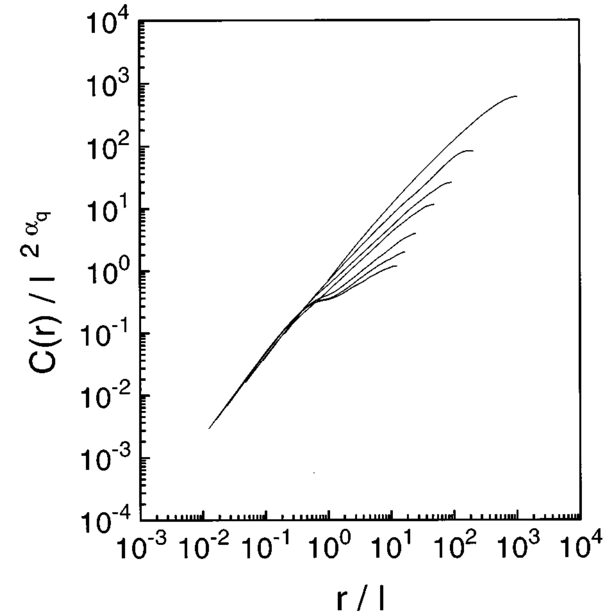

FIG. 7. Double-logarithmic plot of $C(r)/l^{2\alpha_q}$ versus r/l for subset sizes l=5, 11, 21, 41, 61, 81, and 141. The data are for system size $L=2048$.

smaller value than $2\alpha_q$ in Fig. 6 and the numerical estimation of the roughness exponent is likely to be $2\alpha_q - 1$ [Eq. (4)] for fixed l. The behavior of $W^2(L;l) \sim L^{2\alpha_q}$ occurs when l also increases as L increases. In the thermodynamic limit $L \to \infty$, the characteristic size l^* goes to infinity, so that for finite l, the roughness of the overall surface is determined by a random effect and the surface is described in the KPZ universality class. Next, we examine the height-height correlation function

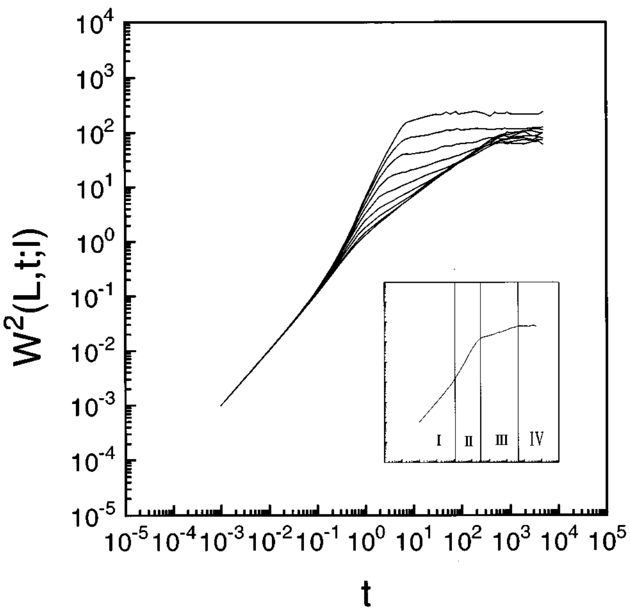

FIG. 8. Double-logarithmic plot of $W^2(L,t;l)$ versus time t for system size $L=1024$. The data are for subset sizes l=3, 5, 9, 17, 33, 65, 129, 257, and 513. Inset: four distinct regimes are observed for the case of $L=1024$ and $l=65$.

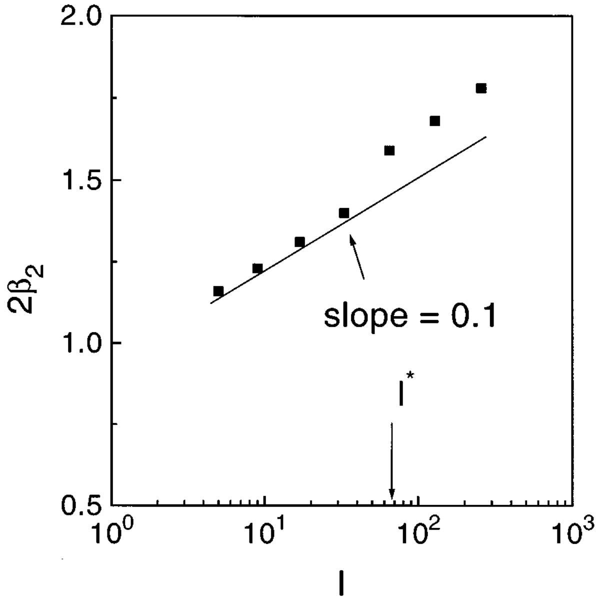

FIG. 9. Semilogarithmic plot of $2\beta_2$ versus l for $L=1024$. The data seem to be on a straight line with slope 0.1 up to the characteristic subset size l^*.

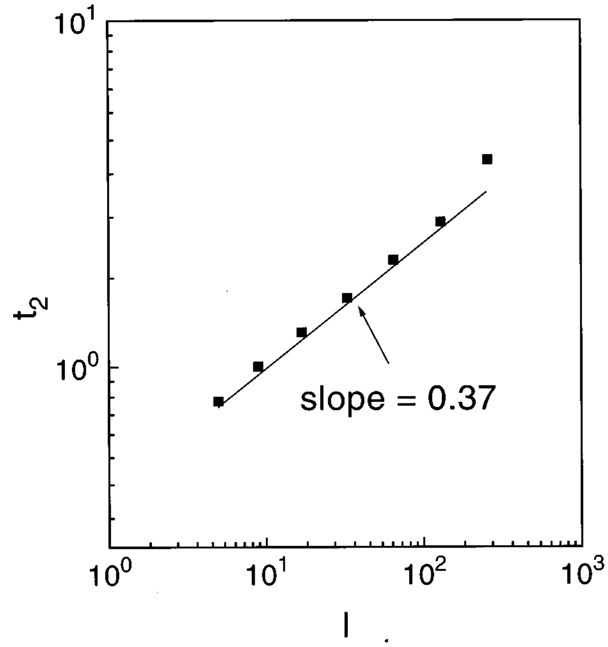

FIG. 10. Double-logarithmic plot of t_2 versus l for $L=1024$. The data seem to be on a straight line with slope 0.37.

$$C(r;L,l) \equiv \left\langle \frac{1}{L} \sum_x [h(x+r)-h(x)]^2 \right\rangle, \tag{6}$$

which is defined for fixed L and l. In Fig. 7 we plot $C(r)/l^{2\alpha_q}$ versus (r/l) in double-logarithmic scales for several values of l and $L=2048$. In Fig. 7 the data are well collapsed for $r/l<1$, while they are not collapsed for $r/l>1$. Accordingly, the coherent surface growth occurs within the range of $r<l$; however, the roughness of the overall surface is determined by the criterion depending on $l^*(L)$ and L above.

It would be interesting to study the l-dependent behavior of dynamic properties of the height fluctuation width $W^2(L,t;l)$. The study is based on numerical simulations for a fixed system size, say, $L=1024$. As shown in the inset of Fig. 8, there exist four distinct regimes for $W^2(L,t;l)$. In the first regime, $W^2(L,t;l)$ increases according to the Poisson distribution and $W^2(L,t;l) \sim t^{2\beta_1}$ with $2\beta_1=1$. The first regime terminates at t_1, which is independent of the subset size l. In the second regime, the surface becomes correlated by a coherent effect, which is caused by the selection of the minimum random number in the selected subset; however, decorrelation also occurs simultaneously by a random effect, which is caused by the selection of the random subset. Since the value of the dynamic exponent $z_s=0.63$ for the Sneppen dynamics is smaller than the one $z_t=1.5$ for the KPZ dynamics, the coherent effect spreads faster than the random effect at early times. Thus the growth exponent $2\beta_2$ in the second regime has a value closer to the Sneppen value $2\beta_s=2$; however, the value is a little smaller due to decorrelation by the random effect. The growth exponent $2\beta_2$ depends on the subset size l as tabulated in Table II. Based on the measurement in Fig. 9, the growth exponent is likely to depend on l as $2\beta_2 \sim (0.1)\ln l$ for $l<l^*$; however, for $l>l^*$, the value of the growth exponent is expected to be close to the one of the Sneppen dynamics as shown in Fig. 8. The second regime terminates at t_2. In Fig. 10 the threshold times are likely to scale as $t_2 \sim l^{0.37}$ for $l<l^*$. The values of the height fluctuation width at the threshold value t_2 are likely to scale as $W_2^2(L,t_2;l) \sim l^{0.92}$, as shown in Fig. 11. In the third regime, the random effect is dominant and the coherence of the surface formed in the second regime becomes decorrelated in this regime. As the subset size is smaller, the third regime is dominant and the growth exponent β_3 becomes much closer

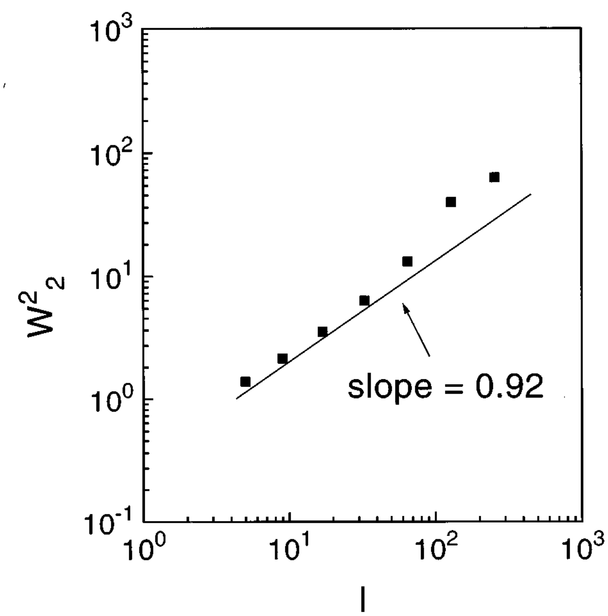

FIG. 11. Double-logarithmic plot of ${W_2}^2$ versus l for $L=1024$. The data seem to be on a straight line with slope 0.92 up to the characteristic subset size l^*.

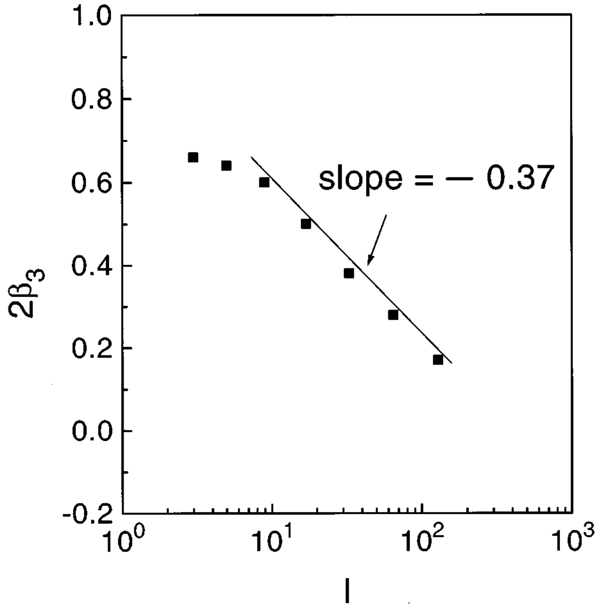

FIG. 12. Semilogarithmic plot of $2\beta_3$ versus l for $L=1024$. The line has the slope -0.37 and the data are for $l=3$, 5, 9, 17, 33, 65, and 129, respectively.

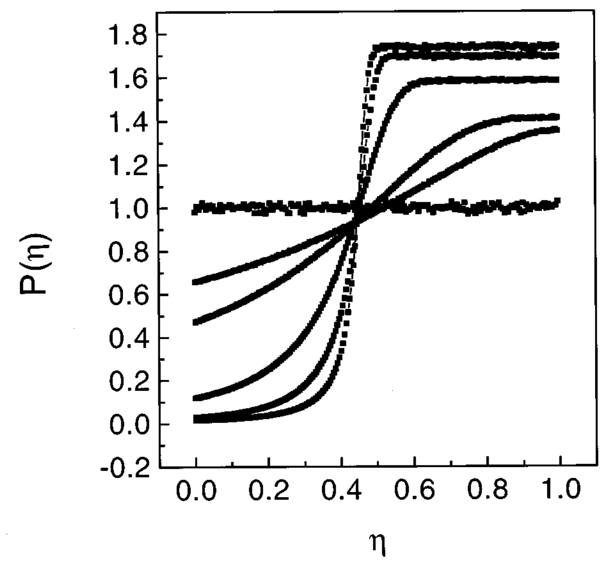

FIG. 13. Plot of the random number distribution after saturation for $L=256$. The data are for $l=1$, 3, 5, 27, 129, and 256, respectively.

to the KPZ value; conversely, as the subset size is larger, the decorrelation effect becomes much weaker, so that the growth exponent β_3 becomes smaller. It is likely that $2\beta_3 \sim (-0.37)\ln l$, as shown in Fig. 12. The third regime terminates at t_3. The numerical values of t_3 for different sizes of l are too close for small l to be measured numerically.

We also investigated the l-dependent behavior of the distribution of random numbers after reaching a saturated state. As shown in Fig. 13, the distribution is flat for $l=1$ and exhibits a critical behavior for $l=L$. Between the two limits, the distributions look like a rounded step function. It would be interesting to note that all distribution functions for different l pass through a specific value of the random number B_c, which corresponds to the threshold of the self-organized critical state [9]. The value of B_c is equal to $1-P_c=0.462$, where P_c is the directed percolation threshold.

Recently, the crossover behavior from the random surface growth to the coherent surface growth was considered by Vergeles [11]. In that study, surface growth occurs at site x on a substrate with the probability $P(x) \sim e^{-q(x)/T}$, where $q(x)$ is a random pinning strength of site x and is also updated with height advance. T is the temperature. It was found that the surface of the model reduces to the one of the Sneppen dynamics when $T=0$; however, for $T \neq 0$, it reduces to the one of the KPZ universality class. The temperature T plays the role of the tuning parameter for the crossover behavior. However, since the tuning parameter is in the form of an exponential function, it is hard to see the finite-size-dependent behavior of the crossover, which is very sensitive to tuning the parameter as we studied in this paper. Nevertheless, the anomalous behavior may barely be observed in the plot of $W(L;T)$ versus L for different temperatures [Fig. 1(b) in Ref. [11]], where the curves of $W(L,T)$ cross each other. However, the crossover behavior was not noted in Ref. [11].

TABLE II. Numerical estimation for the values of the growth exponents, the threshold times, and the height fluctuation widths for various subset sizes l.

l	$2\beta_2$	$\text{Log}_{10}(t_2)$	$\text{Log}_{10}(W_2^2)$	$2\beta_3$
3	1.12	-0.10	0.20	0.66
5	1.16	-0.10	0.20	0.64
9	1.23	0.00	0.33	0.60
17	1.31	0.10	0.55	0.50
33	1.40	0.20	0.80	0.38
65	1.59	0.31	1.12	0.28
129	1.68	0.42	1.60	0.17
257	1.78	0.60	1.80	0.17

In summary, we have introduced a stochastic model for surface growth, which is a generalization of the restricted solid-on-solid model in the KPZ universality class and the Sneppen model in the directed percolation limit, and have investigated the crossover of the two limits. Deposition occurs at the site having a minimum of random numbers within a finite subset rather than of an entire system. The subset is composed of l elements, a randomly selected site, and its $l-1$ neighbors. Changing the subset size l, the height fluctuation width exhibits anomalous behavior with a minimum. The anomalous behavior is due to the two competing effects: the random effect for small l and the coherent effect for large l. The minimum of the surface height fluctuation width, located at $l^* \sim L^{0.59}$, is scaled as $W^2(L;l^*) \sim L^{0.85}$ in 1+1 dimensions. The characteristic subset size $l^*(L)$ has the meaning that for $l<l^*(L)$, the surface grows randomly and belongs to the KPZ universality class, whereas for $l>l^*(L)$, surface grows coherently. The dynamic properties of the crossover have also been investigated. In the early stage of growth, the surface becomes correlated according to the Sneppen dynamics, and in the late stage, the surface correlation is relaxed by the random process. The phenomenon of the dynamic correlation-decorrelation behavior also appears in a stochastic model [12] for the flux line dynamics with transversal and longitudinal fluctuations, which might be described by the coupled quenched KPZ equation [13].

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